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Diffraction of Acoustic Waves by a Semi-Infinite Pipe with Finite Impedance Discontinuity

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Abstract

A rigorous solution is presented for the problem of diffraction of acoustic waves emanating from a ring source by a semi-infinite cylindrical pipe with a partial exterior impedance. By the application of Fourier-transform technique, the diffraction problem is described by a modified Wiener-Hopf equation of the third kind. By performing the Wiener-Hopf factorization and decomposition procedure, the modified Wiener-Hopf equation is reduced to a pair of coupled Fredholm integral equations of the second kind and then solved by iterations.

Key words: Wiener-Hopf technique, Fourier transform, Fredholm integral, diffraction, pipe

1. Introduction

In recent years, air transportation, especially in crowded cities, becomes more important and the number of the airports increase every day. Since, the airports are built nearby the city centers, the people who live next to airports, experience lots of noise pollution. Both the reduction of the people’s live quality and the requirement of the EU standards have made those problems very significant to be solved. Therefore scientists have focused on reducing the scope of sound in modern aircraft jet and turbofan engines, etc. In this context they are investigating the diffraction of acoustic waves by the pipes.

Levine and Schwinger was the first to apply Wiener-Hopf method [1] which is a powerful technique for the diffraction problem, to the study of sound radiation from an unflanged rigid cylindrical duct [2]. For lined pipes, scientific studies show that there is a potential benefit of an acoustic liners for noise reduction. Rawlins who first obtained the exact solution for the problem of radiation of sound waves from a semi-infinite rigid duct with an acoustically absorbing internal surface [3]. He proved that acoustic liners are one of the effective method of reducing noise from ducts. Demir and Buyukaksoy solved the same problem with partial lining [4]. In their work, a solution involves a set of infinitely many coefficients satisfying an infinite system of linear algebraic equation. The effect of parameters, such as pipe radius, internal surface impedance, etc. on the diffracted phenomenon were shown graphically. Later, Buyukaksoy and Polat considered the diffraction of acoustic waves by a semi-infinite cylindrical impedance pipe of certain wall thickness [5]. They analyzed the effect of different linings from inside, outside, end side and wall thickness with graphics. In both studies above, a hybrid method was applied. Then, Tiryakioglu and Demir examined the problem of diffraction of waves from a semi-infinite

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rigid cylindrical duct with external impedance surface [6]. Here the ring source which is located out of the pipe was used to illuminate the cylindrical duct. An analytical solution is derived by solving the Wiener-Hopf equation.

In this paper the diffraction of acoustic waves emanating from a ring source by a semi-infinite cylindrical pipe with a partial lining on the outer surface is investigated rigorously through the Wiener-Hopf technique. The pipe which walls are assumed to be infinitely thin, represents the nozzle of the jet engine and illuminating nozzle by a ring source. By the application of the Fourier transform technique, the related boundary value problem is formulated as a modified Wiener-Hopf equation of the third kind and then reduced to a pair of simultaneous Fredholm integral equations of the second kind which are susceptible to a treatment by iterations. The solution involves branch-cut integrals with unknown integrands which have to be performed numerically.

2. Analysis

2.1. Formulation of the Problem

We consider the diffraction of acoustic waves by a semi-infinite cylindrical pipe. The geometry of the problem is considered as a mathematical model of aircraft engine (Figure 1).

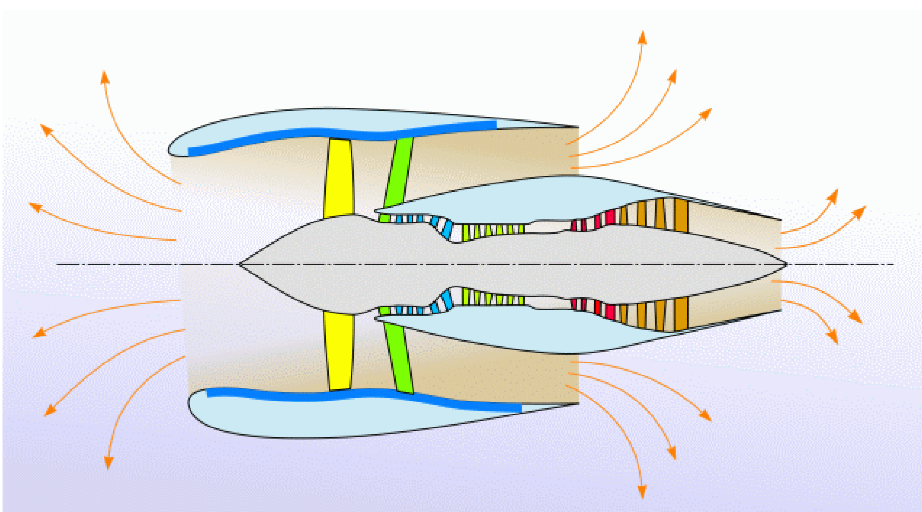

Figure 1. Sketch of a turbofan aero-engine

The pipe walls are assumed to be infinitely thin and they occupy the region $\{r = a, z \in (-\infty, l)\}$ illuminated by a ring source located at $r = b > a, z = 0$ (Figure 2). The part $r = a, z \in (0, l)$ of its exterior surface is assumed to be treated by an acoustically absorbent lining which is denoted by Z, while the other parts of the pipe are assumed to be rigid. We introduce a scalar potential $\psi(r, z, t)$ which defines the acoustic pressure and velocity by $p = -\rho_0(\partial/\partial t)\psi$ and $\vec{v} = grad\psi$ respectively, where ρ_0 is the density of the undisturbed medium.

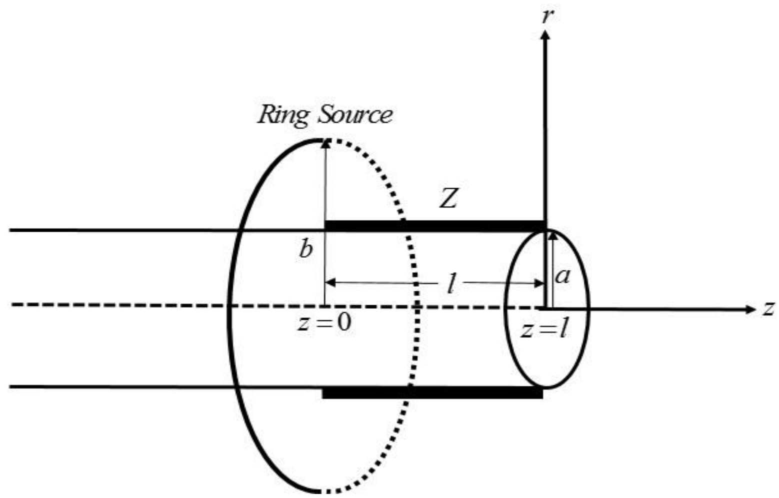

Figure 2. Geometry of the problem

From the symmetry of the geometry of the problem and of the ring source, the total field will be independent of θ everywhere in circular cylindrical coordinate system (r, θ, z).

For analysis purposes, it is convenient to express the total field as follows:

$$\psi^T(r,z,t) = \begin{cases} \psi_1(r,z)exp(-i\omega t) & , \quad r > b \quad , \ z \in (-\infty,\infty) \\ \psi_2(r,z)exp(-i\omega t) & , \ r \in (a,b) \ , \ z \in (-\infty,\infty) \\ \psi_3(r,z)exp(-i\omega t) & , \ r \in (0,a) \ , \ z \in (-\infty,\infty) \end{cases} \tag{1}$$

where $\omega = 2\pi f$ is the angular frequency. Time dependence is assumed to be $e^{-i\omega t}$ and suppressed throughout this work.

2.2. Reduction to a Modified Wiener-Hopf Equation

The unknown velocity potentials $\psi_1(r,z), \psi_2(r,z)$ and $\psi_3(r,z)$ satisfy the Helmholtz equation for $z \in (-\infty,\infty)$.

$$\left[\frac{1}{r}\frac{\partial}{\partial \mathrm{r}}\left(r\frac{\partial}{\partial \mathrm{r}}\right) + \frac{\partial^2}{\partial \mathrm{z}^2} + \mathrm{k}^2\right]\psi_{1,2,3}(r,z) = 0 \tag{2}$$

By taking Fourier transform of these equations, one can obtain the following integral forms:

$$\psi_1(r,z) = \frac{k}{2\pi}\int_L A(u)H_0^{(1)}(\lambda kr)e^{-iukz}du \tag{3}$$

$$\psi_2(r,z) = \frac{k}{2\pi}\int_L [B(u)J_0(\lambda kr) + C(u)Y_0(\lambda kr)]e^{-iukz}du \tag{4}$$

$$\psi_3(r,z) = \frac{k}{2\pi}\int_L D(u)J_0(\lambda kr)e^{-iukz}du \tag{5}$$

where L is a suitable inverse Fourier transform integration contour along or near to the real axis in the complex $u-$ domain (Figure 3).

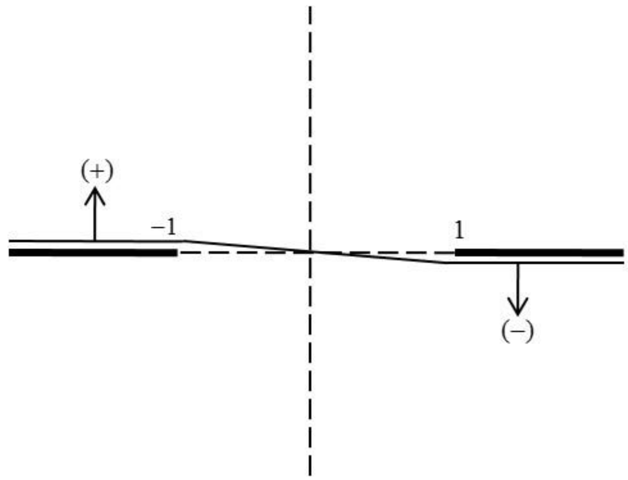

Figure 3. Complex $u-$ plane with Fourier contour and branch cut

J_0 and Y_0 are the Bessel and Neumann functions of order zero, $H_0^{(1)} = J_0 + iY_0$ is the Hankel function of the first type. λ is a square root function which is defined as

$$\lambda(u) = \sqrt{1-u^2} \tag{6}$$

Branch cuts for λ is taken on the line from 1 to ∞ and from $-\infty$ to -1. We will assume that the surrounding medium is slightly lossy and k has small positive part. The lossless case can be obtained by letting $Imk \to 0$ at the end of the analysis.

From the following boundary conditions and relations of continuity, $A(u), B(u), C(u)$ and $D(u)$ will be obtained.

$$\frac{\partial}{\partial r}\psi_1(b,z) - \frac{\partial}{\partial r}\psi_2(b,z) = \delta(z-0) \quad , \quad z \in (-\infty,\infty) \tag{7}$$

$$\psi_1(b,z) = \psi_2(b,z) \quad , \quad z \in (-\infty,\infty) \tag{8}$$

$$\frac{\partial}{\partial r}\psi_2(a,z) = \frac{ik}{Z}\psi_2(a,z) \quad , \quad 0 < z < l \tag{9}$$

$$\frac{\partial}{\partial r}\psi_2(a,z) = 0 \quad , \quad z < 0 \tag{10}$$

$$\frac{\partial}{\partial r}\psi_3(a,z) = 0 \quad , \quad z < l \tag{11}$$

$$\frac{\partial}{\partial r}\psi_2(a,z) = \frac{\partial}{\partial r}\psi_3(a,z) \,, \quad z > l \tag{12}$$

$$\psi_2(a,z) = \psi_3(a,z) \,, \quad z > l \tag{13}$$

The spectral coefficients $A(u), B(u)\ and\ C(u)$ are related to each other by the definition of the ring source given in (7-8), the Fourier transform which give,

$$\lambda k A(u) H_1^{(1)}(\lambda k b) = \lambda k B(u) J_1(\lambda k b) + \lambda k C(u) Y_1(\lambda k b) - 1 \tag{14}$$

$$A(u) H_0^{(1)}(\lambda k b) = B(u) J_0(\lambda k b) + C(u) Y_0(\lambda k b) \tag{15}$$

The elimination of $C(u)$ between (14) and (15), then the elimination of $B(u)$, we can obtain the following coefficients

$$B(u) = A(u) + \frac{\pi b}{2} Y_0(\lambda k b) \tag{16}$$

$$C(u) = iA(u) - \frac{\pi b}{2} J_0(\lambda k b) \tag{17}$$

Applying the boundary condition on $r = a$, from (10) and (11)

$$-B(u)\lambda k J_1(\lambda k a) - C(u)\lambda k Y_1(\lambda k a) = \frac{ik}{Z} T(u) + e^{iukl}\phi_1^+(u) \tag{18}$$

$$-D(u)\lambda k J_1(\lambda k a) = e^{iukl}\phi_1^+(u) \tag{19}$$

Continuity relations at $r = a$ yields

$$D(u) J_0(\lambda k a) - B(u) J_0(\lambda k a) - C(u) Y_0(\lambda k a) = e^{iukl}\phi_1^-(u) \tag{20}$$

where $\phi_1^+(u)\ and\ \phi_1^-(u)$ are analytic functions in the upper half plane $(Imu > 0\ or\ Imu = 0\ and\ Reu > 0)$ and in the lower half plane $(Imu < 0\ or\ Imu = 0\ and\ Reu < 0)$, respectively, while $T(u)$ is an entire function.

$$\phi_1^+(u) = \int_l^{\infty} \frac{\partial}{\partial r}\psi_2(a,z)\, e^{iuk(z-l)} dz \tag{21}$$

$$\phi_1^-(u) = \int_{-\infty}^{l} [\psi_3(a,z) - \psi_2(a,z)]\, e^{iuk(z-l)} dz \tag{22}$$

$$T(u) = \int_{0}^{l} \psi_2(a,z) e^{iukz} dz \tag{23}$$

From (18) we obtain

$$A(u) = -\frac{1}{\lambda k H_1^{(1)}(\lambda k a)} \left\{ \frac{ik}{Z} T(u) + e^{iukl}\phi_1^+(u) + \frac{\lambda k \pi b}{2} [J_1(\lambda k a) Y_0(\lambda k b) - J_0(\lambda k b) Y_1(\lambda k a)] \right\} \tag{24}$$

By substituting $B(u), C(u)$ and $D(u)$ in (20) we get the following Modified Wiener-Hopf Equation (MWHE) of the third kind.

$$\phi_1^+(u) M(u) + T(u) N(u) e^{-iukl} + \frac{b}{a} \frac{H_0^{(1)}(\lambda k b)}{\lambda k H_1^{(1)}(\lambda k b)} e^{-iukl} = \phi_1^-(u) \tag{25}$$

where

$$M(u) = \frac{H_0^{(1)}(\lambda k a)}{\lambda k H_1^{(1)}(\lambda k a)} - \frac{J_0(\lambda k a)}{\lambda k J_1(\lambda k a)} \tag{26}$$

$$N(u) = \frac{(i/Z) H_0^{(1)}(\lambda k a)}{\lambda H_1^{(1)}(\lambda k a)} \tag{27}$$

2.3. Approximate Solution of the Modified Wiener-Hopf Equation

By using the factorization and decomposition procedures, together with the Liouville theorem, the modified Wiener-Hopf equation in (25) can be reduced to the following system of Fredholm integral equations of the second kind:

$$\phi_1^+(u) M_+(u) = -\frac{1}{2\pi i} \frac{b}{a} \int_{L_+} \frac{H_0^{(1)}(\lambda k b) M_-(\tau) e^{-i\tau k l}}{\lambda k H_1^{(1)}(\lambda k a)(\tau - u)} d\tau - \frac{1}{2\pi i} \int_{L_+} \frac{T(\tau) N(\tau) M_-(\tau) e^{-i\tau k l}}{\tau - u} d\tau \tag{28}$$

$$\frac{T(u)}{N_-(u)} e^{-iukl} = \frac{1}{2\pi i} \frac{b}{a} \int_{L_-} \frac{H_0^{(1)}(\lambda k b) e^{-i\tau k l}}{\lambda k H_1^{(1)}(\lambda k a) N_+(\tau)(\tau - u)} d\tau + \frac{1}{2\pi i} \int_{L_-} \frac{\phi_1^+(\tau) M(\tau)}{N_+(\tau)(\tau - u)} d\tau$$

$$-\frac{1}{2\pi i}\int_{L_-}\frac{\phi_1^-(\tau)}{N_+(\tau)(\tau-u)}d\tau \qquad (29)$$

$$\phi_1^-(\tau) = -\frac{1}{2\pi i}\frac{b}{a}\int_{L_-}\frac{H_0^{(1)}(\lambda kb)e^{-i\tau kl}}{\lambda kH_1^{(1)}(\lambda ka)(\tau-u)}d\tau - \frac{1}{2\pi i}\int_{L_-}\frac{\phi_1^+(\tau)M(\tau)}{\tau-u}d\tau$$
$$-\frac{1}{2\pi i}\int_{L_-}\frac{T(\tau)N(\tau)e^{-i\tau kl}}{\tau-u}d\tau \qquad (30)$$

where $M_+(u),\ N_+(u)$ and $M_-(u),\ N_-(u)$ are the split functions, analytic and free of zeros in the upper and lower halves of the complex $u-$plane, respectively, resulting from the Wiener-Hopf factorization of $M(u)$ and $N(u)$ which are given by (26) and (27), in the following form:

$$M(u) = \frac{M_+(u)}{M_-(u)} \quad , \quad N(u) = \frac{N_+(u)}{N_-(u)} \qquad (31,32)$$

Here the explicit forms for $M_+(u),\ M_-(u)$ and $N_+(u),\ N_-(u)$ can be obtained as is done in [7] and they will be calculated numerically. For large argument, the coupled system of Fredholm integral equations of the second kind in (28) - (30), is susceptible to a treatment by iterations.

$$\phi_1^+(u) = \phi_{1,1}^+(u) + \phi_{1,2}^+(u) + \cdots \qquad (33)$$

$$\phi_1^-(u) = \phi_{1,1}^-(u) + \phi_{1,2}^-(u) + \cdots \qquad (34)$$

$$T(u) = T_1(u) + T_2(u) + \cdots \qquad (35)$$

From the first iterations, we get

$$\phi_{1,1}^+(u)M_+(u) = -\frac{1}{2\pi i}\frac{b}{a}\int_{L_+}\frac{H_0^{(1)}(\lambda kb)M_-(\tau)e^{-i\tau kl}}{\lambda kH_1^{(1)}(\lambda ka)(\tau-u)}d\tau = I_1(u) \qquad (36)$$

$$\frac{T_1(u)}{N_-(u)}e^{-iukl} = \frac{1}{2\pi i}\frac{b}{a}\int_{L_-}\frac{H_0^{(1)}(\lambda kb)e^{-i\tau kl}}{\lambda kH_1^{(1)}(\lambda ka)N_+(\tau)(\tau-u)}d\tau = I_2(u) \qquad (37)$$

and

$$\phi_{1,1}^-(\tau) = -\frac{1}{2\pi i}\frac{b}{a}\int_{L_-}\frac{H_0^{(1)}(\lambda kb)e^{-i\tau kl}}{\lambda kH_1^{(1)}(\lambda ka)(\tau-u)}d\tau = I_3(u) \qquad (38)$$

while the second iteration gives

$$\phi_{1,2}^{+}(u)M_{+}(u) = -\frac{1}{2\pi i}\int_{L_+} \frac{I_2(\tau)N_-(\tau)M_-(\tau)N(\tau)}{\tau - u}d\tau = J_1(u) \tag{39}$$

and

$$\frac{T_2(u)}{N_-(u)}e^{-iukl} = \frac{1}{2\pi i}\int_{L_-} \frac{I_1(\tau)M(\tau)}{M_+(\tau)N_+(\tau)(\tau - u)}d\tau - \frac{1}{2\pi i}\int_{L_-} \frac{I_3(u)}{N(\tau)N_-(\tau)(\tau - u)}d\tau = J_2(u) \tag{40}$$

Now, the approximate solution of the MWHE reads:

$$\phi_1^{+}(u)M_{+}(u) = I_1(u) + J_1(u) \tag{41}$$

$$\frac{T(u)}{N_-(u)}e^{-iukl} = I_2(u) + J_2(u) \tag{42}$$

The integral given by (36) can be evaluated by means of the steepest-descent method. Using the asymptotic expansion of Hankel's functions for large argument and making the following substitutions, one can easily obtained the integral.

$$\tau = -cos\xi \quad , \quad b - a = R_0 sin\theta_0 \quad , \quad l = R_0 cos\theta_0 \tag{43}$$

Similar way is valid for $I_2(u)$ and $I_3(u)$ which are given by (37), (38). For (39) and (40), these integrals can only be obtained numerically. First the integration line $L_{\pm}$ can be deformed onto the branch-cut, then the integrals can be evaluated by the help of Cauchy theorem.

3. Far Field

The total field in the region $r > b$ can be obtained from (3) and (24)

$$\psi_1(r,z) = -\frac{1}{2\pi}\int_L \frac{\phi_1^{+}(u)}{\lambda H_1^{(1)}(\lambda ka)}H_0^{(1)}(\lambda kr)e^{-iuk(z-l)}du$$
$$-\frac{k}{2\pi}\int_L \left[\frac{(i/Z)T(u)}{\lambda H_1^{(1)}(\lambda ka)} + \frac{\pi b}{2H_1^{(1)}(\lambda ka)}[J_1(\lambda ka)Y_0(\lambda kb) - J_0(\lambda kb)Y_1(\lambda ka)]\right]H_0^{(1)}(\lambda kr)e^{-iukz}du \tag{44}$$

where L is the inverse Fourier transform contour. Utilizing the asymptotic expansion of $H_0^{(1)}(\lambda kr)$ valid for $kr \gg 1$

$$H_0^{(1)}(\lambda kr) \sim \sqrt{\frac{2}{\pi\lambda kr}}e^{i(\lambda kr - \pi/4)} \tag{45}$$

and using the saddle point technique, we get

$$\psi_1(r,z) \sim \frac{i}{\pi}\left[\frac{\phi_1^+(-\cos\theta_1)}{\sin\theta_1\, H_1^{(1)}(ka\sin\theta_1)}\right]\frac{e^{ikR_1}}{kR_1} + \frac{i}{\pi}\left[\frac{(ik/Z)T(-\cos\theta_2)}{\sin\theta_2\, H_1^{(1)}(ka\sin\theta_2)} - \frac{\pi kb}{2H_1^{(1)}(ka\sin\theta_2)}\right.$$
$$\left.\times\left[J_1(ka\sin\theta_2)Y_0(kb\sin\theta_2) - J_0(kb\sin\theta_2)Y_1(ka\sin\theta_2)\right]\right]\frac{e^{ikR_2}}{kR_2} \quad (46)$$

where R_1, θ_1 and R_2, θ_2 are the spherical coordinates defined by

$$r = R_1 sin\theta_1 \quad , \quad \mathrm{z} - \mathrm{l} = R_1 cos\theta_1 \quad (47)$$

and

$$r = R_2 sin\theta_2 \quad , \quad \mathrm{z} = R_2 cos\theta_2 \quad (48)$$

as shown in Figure 4.

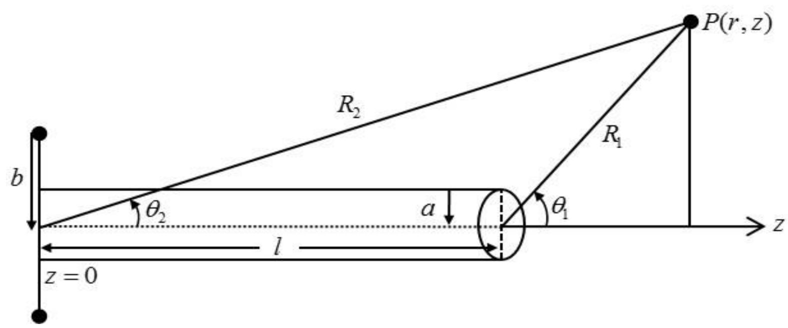

Figure 4. Spherical coordinates

4. Conclusions

This study analyzes the diffraction of acoustic waves emanating from a ring source by a rigid semi-infinite pipe whose outer surface is treated by an acoustically absorbing material of finite length. This problem is more complicated due to partial lining of the exterior surface. To overcome the additional difficulty caused by the finite impedance discontinuity, the problem was first reduced to a system of Fredholm integral equations of the second kind and then solved approximately by iterations.

In forthcoming study, to a better understanding the effect of partial lining on the diffracted field, numerical calculations and graphics are going to be obtained.

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